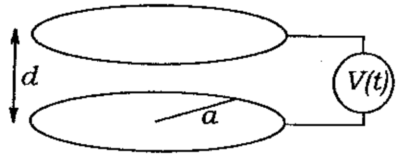
Exercise 11-2.



CHAPTER 11

(a) The electric field between the plates, in the quasistatic approximation, is

$$\mathbf{E} = \frac{V(t)}{d}\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is normal to the plates. By the Ampère-Maxwell equation, the magnetic field between the plates is

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{\mu_0 \epsilon_0}{d}\frac{dV}{dt}\hat{\mathbf{n}}.$$

Consider an Amperian loop coaxial with the plates, of radius r with $r > a$. By symmetry $\mathbf{B}$ is azimuthal. Therefore $\oint \mathbf{B} \cdot d\boldsymbol{\ell} = 2\pi r B_\phi$. Also, by Stokes's theorem,

$$\oint \mathbf{B} \cdot d\boldsymbol{\ell} = \int \nabla \times \mathbf{B} \cdot d\mathbf{A} = \frac{\mu_0 \epsilon_0}{d}\frac{dV}{dt}\pi a^2.$$

Equating the two expressions for the circulation of $\mathbf{B}$, we find

$$B_\phi = \frac{\mu_0 \epsilon_0 a^2}{2rd}\frac{dV}{dt}.$$

(b) The charge on the plates is $\pm Q$ where

$$Q = CV = \frac{\epsilon_0 A}{d}V = \frac{\pi a^2 \epsilon_0}{d}V,$$

and the current between the plates is

$$I = \frac{dQ}{dt} = \frac{\pi a^2 \epsilon_0}{d}\frac{dV}{dt}.$$

The magnetic field due to the same current I in a wire is $\mathbf{B} = B_\phi \hat{\boldsymbol{\phi}}$ where

$$B_\phi = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 \epsilon_0 a^2}{2rd}\frac{dV}{dt}.$$

This is the same as the result found in (a).

Exercise 11-7. For a uniform linear medium, the energy density is $u = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$ and the energy flux density is $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, where $\mathbf{D} = \epsilon \mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu$. The divergence of $\mathbf{S}$ is

$$\begin{aligned}
\nabla \cdot \mathbf{S} &= \epsilon_{ijk}\partial_i (E_j H_k) \\
&= (\nabla \times \mathbf{E}) \cdot \mathbf{H} - (\nabla \times \mathbf{H}) \cdot \mathbf{E}
\end{aligned}$$

$$\begin{aligned}
&= -\frac{\partial \mathbf{B}}{\partial t} \cdot \frac{\mathbf{B}}{\mu} - \left(\mathbf{J}_f + \epsilon \frac{\partial \mathbf{E}}{\partial t}\right) \cdot \mathbf{E} \\
&= -\frac{\partial}{\partial t}\left\{\frac{1}{2}(\mathbf{B}\cdot\mathbf{H} + \mathbf{E}\cdot\mathbf{D})\right\} - \mathbf{J}_f \cdot \mathbf{E} \\
&= -\frac{\partial u}{\partial t} - \frac{\partial u_K}{\partial t}.
\end{aligned}$$

We identify that the rate of energy transfer to the material is

$$\frac{\partial u_K}{\partial t} = \mathbf{J}_f \cdot \mathbf{E}.$$

This is the rate that the electric field does work on the free charge.

Exercise 11-9. The specific plane wave solution described in Sec. 11.5.2 is

$$\begin{aligned} \mathbf{E}(\mathbf{x},t) &= E_0\hat{\mathbf{i}}e^{i(kz-\omega t)} \\ \mathbf{B}(\mathbf{x},t) &= B_0\hat{\mathbf{j}}e^{i(kz-\omega t)} \end{aligned}$$

where $\omega = ck$ and $B_0 = E_0/c$. Now verify Maxwell's equations. There are no charges or currents: These fields describe an electromagnetic wave in vacuum.

- Gauss's law is satisfied for both **E** and **B**:

$$\begin{aligned} \nabla\cdot\mathbf{E} &= \frac{\partial E_x}{\partial x} = 0 \\ \nabla\cdot\mathbf{B} &= \frac{\partial B_y}{\partial y} = 0 \end{aligned}$$

- Faraday's law is satisfied:

$$\begin{aligned} \nabla\times\mathbf{E} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \partial_z \\ E_x & 0 & 0 \end{vmatrix} = \hat{\mathbf{j}}\frac{\partial E_x}{\partial z} = ikE_0e^{i(kz-\omega t)}\hat{\mathbf{j}} \\ -\frac{\partial\mathbf{B}}{\partial t} &= i\omega B_0e^{i(kz-\omega t)}\hat{\mathbf{j}} \end{aligned}$$

These are equal because

$$\omega B_0 = ck\frac{E_0}{c} = kE_0.$$

- Ampère's law is satisfied:

$$\begin{aligned} \nabla\times\mathbf{B} &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 0 & 0 & \partial_z \\ 0 & B_y & 0 \end{vmatrix} = -\hat{\mathbf{i}}\frac{\partial B_y}{\partial z} = -ikB_0e^{i(kz-\omega t)}\hat{\mathbf{i}} \\ \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} &= \frac{-i\omega}{c^2}E_0e^{i(kz-\omega t)}\hat{\mathbf{i}} \end{aligned}$$

These are equal because

$$\frac{\omega}{c^2}E_0 = \frac{ck}{c^2}cB_0 = kB_0.$$

Exercise 11-14. (a) For a plane wave, $B_0 = E_0/c$. Therefore,

$$B_0 = \frac{5\times10^{-3}\,\text{V/m}}{3\times10^8\,\text{m/s}} = 1.7\times10^{-11}\,\text{T}.$$

(b) The intensity of a plane wave is

$$\mathcal{I} = \bar{u}c = c\epsilon_0E_0^2/2.$$

Therefore,

$$\begin{aligned} \mathcal{I} &= \frac{1}{2}\left(3\times10^8\,\text{m/s}\right)\left(8.85\times10^{-12}\,\text{C}^2\text{N}^{-1}\text{m}^{-2}\right)\left(5\times10^{-3}\,\text{Vm}^{-1}\right)^2 \\ &= 3.3\times10^{-8}\,\text{Wm}^{-2}. \end{aligned}$$

Exercise 11-17. The electric field is

$$\mathbf{E}(\mathbf{x},t)=\frac{E_0}{\sqrt{2}}\left(\hat{\mathbf{k}}-\hat{\mathbf{i}}\right)\sin\left(ky-\omega t\right).$$

Note that $\mathbf{E}$ is orthogonal to the direction of propagation, $\hat{\mathbf{j}}$.

(a) Use Faraday's law to determine the magnetic field:

$$\begin{aligned}\frac{\partial \mathbf{B}}{\partial t} &= -\nabla\times\mathbf{E}=-\begin{vmatrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 0 & \partial_y & 0\\ E_x & 0 & E_z\end{vmatrix}\\ &= -\hat{\mathbf{i}}\frac{\partial E_z}{\partial y}+\hat{\mathbf{k}}\frac{\partial E_x}{\partial y}\\ &= \frac{E_0 k}{\sqrt{2}}\left(-\hat{\mathbf{i}}-\hat{\mathbf{k}}\right)\cos\left(ky-\omega t\right).\end{aligned}$$

Thus

$$\mathbf{B}=\frac{E_0}{\sqrt{2}c}\left(\hat{\mathbf{i}}+\hat{\mathbf{k}}\right)\sin\left(ky-\omega t\right)$$

where we have used the dispersion relation $\omega=ck$. Note that $\mathbf{E}$, $\mathbf{B}$ and $\hat{\mathbf{j}}$ form an orthogonal triad, a fact that could have been used to determine $\mathbf{B}$ without any differential equation.

(b) The Poynting vector is

$$\begin{aligned}\mathbf{S} &= \frac{\mathbf{E}\times\mathbf{B}}{\mu_0}=\frac{E_0^2}{2\mu_0 c}\left(\hat{\mathbf{k}}-\hat{\mathbf{i}}\right)\times\left(\hat{\mathbf{k}}+\hat{\mathbf{i}}\right)\sin^2(ky-\omega t)\\ &= \frac{E_0^2}{\mu_0 c}\hat{\mathbf{j}}\sin^2(ky-\omega t).\end{aligned}$$

Exercise 11-21. The mean intensity (=energy flux averaged over a period of oscillation) of a plane wave is

$$\mathcal{I}=S_{\text{avg}}=u_{\text{avg}}c=\frac{1}{2}\epsilon_0 E_0^2 c=\epsilon_0 c E_{\text{rms}}^2.$$

Note that $\epsilon_0=8.85\times10^{-12}\,\mathrm{J\,m^{-1}\,V^{-2}}$.

(a) For sunlight at the Earth:

$$\begin{aligned}E_{\text{rms}} &= \sqrt{\frac{1300\,\mathrm{W/m^2}}{\epsilon_0 c}}=700\,\mathrm{V/m},\\ B_{\text{rms}} &= \frac{E_{\text{rms}}}{c}=2.3\times10^{-6}\,\mathrm{T}.\end{aligned}$$

The RMS fields do not depend on the wavelength.

(b) At a distance of 1 m from a 100 W bulb:

$$\begin{aligned}S_{\text{avg}} &= \frac{P}{4\pi r^2}=\frac{100\,\mathrm{W}}{4\pi(1\,\mathrm{m})^2}=7.96\,\mathrm{W/m^2},\\ E_{\text{rms}} &= \sqrt{\frac{S_{\text{avg}}}{\epsilon_0 c}}=55\,\mathrm{V/m}.\end{aligned}$$

(c) For the laser pointer beam of diameter $d=4\,\mathrm{mm}$:

$$S_{\text{avg}}=\frac{P}{\pi(d/2)^2}=\frac{0.1\,\mathrm{mW}}{\pi(2\,\mathrm{mm})^2}=8.0\,\mathrm{W/m^2},$$

$$E_{\text{rms}}=\sqrt{\frac{S_{\text{avg}}}{\epsilon_0 c}}=55\,\mathrm{V/m}.$$

(d) The energy of a photon is $h\nu=hc/\lambda$, so the photon emission rate is

$$\frac{dN}{dt}=\frac{P}{h\nu}=\frac{P\lambda}{hc}=\frac{(0.1\,\mathrm{mW})(633\,\mathrm{nm})}{(6.63\times10^{-34}\,\mathrm{Js})(3\times10^{8}\,\mathrm{m/s})}=3\times10^{14}\text{ photons per second.}$$