with $E_a < V_0$ and $E_b > V_0$.

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APPENDIX A: TRIGONOMETRIC INTEGRALS

For many of the calculations included here, we require versions of the integral

$$\int_{-\infty}^{+\infty} \frac{\sin(z)\ \cos(mz)}{z}\, dz = \begin{cases} 0 & \text{for } m < -1 \text{ and } m > +1 \\ \pi/2 & \text{for } m = \pm 1 \\ \pi & \text{for } m^2 < 1 \end{cases} \tag{A1}$$

which is a handbook [47] result. We can briefly justify these results, starting with the single integral

$$\int_{-\infty}^{+\infty} \frac{\sin(z)}{z}\, dz = \pi \tag{A2}$$

which itself can be derived using contour integration. This can be generalized to

$$\int_{-\infty}^{+\infty} \frac{\sin(mz)}{z}\, dz = \begin{cases} +\pi & \text{for } m > 0 \\ 0 & \text{for } m = 0 \\ -\pi & \text{for } m < 0 \end{cases} \tag{A3}$$

by a change of variables, and considering the special $m = 0$ case separately. The general integral in Eqn. (A1) can then be obtained by writing

$$\sin(z)\cos(mz) = \frac{1}{2}\left\{\sin[(1+m)z] + \sin[(1-m)z]\right\} \tag{A4}$$

and using Eqn. (A3) twice. The special case of $m = 1$ is done by noting that

$$\int_{-\infty}^{+\infty} \frac{\sin(z)\ \cos(z)}{z}\, dz = \int_{-\infty}^{+\infty} \frac{\sin(2z)}{2z}\, dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{\sin(w)}{w}\, dw = \frac{\pi}{2} \tag{A5}$$

where $w = 2z$ and a simple trig identity is used. Another integral that is useful for normalization calculations is

$$\int_{-\infty}^{+\infty} \frac{\sin^2(z)}{z^2}\, dz = \pi \tag{A6}$$

which is another handbook result derivable using contour integration techniques.

APPENDIX B: WIGNER DISTRIBUTION FROM MOMENTUM-SPACE WAVEFUNCTIONS

The evaluation of the Wigner function, using momentum-space wavefunctions as in Eqn. (5), requires the integral

$$\begin{aligned} P_W^{(n)}(x,p) \;=\; & \left(\frac{L}{(\pi\hbar)^2}\right) \int_{-\infty}^{+\infty} dq\, e^{-2iqx/\hbar}\, e^{+i(p+q)L/2\hbar}\, e^{-i(p-q)L/2\hbar} \\ & \times \left\{ e^{-in\pi/2} \frac{\sin[((p+q)L/\hbar - n\pi)/2]}{[(p+q)L/\hbar - n\pi]} - e^{+in\pi/2} \frac{\sin[((p+q)L/\hbar + n\pi)/2]}{[(p+q)L/\hbar + n\pi]} \right\} \\ & \times \left\{ e^{+in\pi/2} \frac{\sin[((p-q)L/\hbar - n\pi)/2]}{[(p-q)L/\hbar - n\pi]} - e^{-in\pi/2} \frac{\sin[((p-q)L/\hbar + n\pi)/2]}{[(p-q)L/\hbar + n\pi]} \right\} \end{aligned} \tag{B1}$$

and we briefly sketch out some of the necessary steps in the evaluation of $P_W^{(n)}(x,p)$ in this approach.

Combining the various complex exponentials and using the fact that the Wigner function must be real, we are left with integrals such as

$$\mathcal{I}_1 = \int_{-\infty}^{+\infty} \cos\left[\frac{qL}{\hbar}\left(\frac{L-2x}{L}\right)\right] \frac{\sin[((p+q)L/\hbar - n\pi)/2]}{[(p+q)L/\hbar - n\pi]} \frac{\sin[((p-q)L/\hbar - n\pi)/2]}{[(p-q)L/\hbar - n\pi]}\, dq\,. \tag{B2}$$

In this case, the appropriate partial fraction identity to rewrite the denominators is

$$\begin{aligned} \frac{1}{((p+q)L/\hbar - n\pi)((p-q)L/\hbar - n\pi)} \;=\; & \frac{1}{2(pL/\hbar - n\pi)} \left\{ \frac{1}{((p+q)L/\hbar - n\pi)} \right. \\ & \left. + \frac{1}{((p-q)L/\hbar - n\pi)} \right\} \end{aligned} \tag{B3}$$

and the remaining integrals can be done using variations on Eqn. (41). Upon combining various factors, one obtains integrals of that form not only with $m = (L-2x)/L$, as in Eqn. (40), but similar ones with $m = (L-4x)/L$ and $m = (3L-4x)/L$. These terms give rise to limits on the x dependence of the form $\mathcal{R}(x;0,L/2)$ and $\mathcal{R}(x;L/2,L)$ where $\mathcal{R}(x;a,b)$ is defined in Eqn. (45). For example, one intermediate result can be written in the form

$$\begin{aligned} \mathcal{T} \;=\; & \cos\left[\left(\frac{pL}{\hbar} - n\pi\right)\left(\frac{L-2x}{L}\right)\right] \sin\left[\left(\frac{pL}{\hbar} - n\pi\right)\right] \mathcal{R}(x;0,L) \\ & - \sin\left[\left(\frac{pL}{\hbar} - n\pi\right)\left(\frac{L-2x}{L}\right)\right] \cos\left[\left(\frac{pL}{\hbar} - n\pi\right)\right] \{\mathcal{R}(x;0,L/2) - \mathcal{R}(x;L/2,L)\} \end{aligned} \tag{B4}$$

which, in turn, gives

$$\mathcal{T} = \begin{cases} 0 & \text{for } x < 0, x > L \\ \sin[(2(p/\hbar - n\pi/L)x] & \text{for } 0 < x < L \\ \sin[(2(p/\hbar - n\pi/L)(L-x)] & \text{for } L/2 < x < L \end{cases} \tag{B5}$$

so that the 'split' definition of $P_W^{(n)}(x,p)$ in the two half intervals arises very naturally and the complete result in Eqn. (60) is reproduced using momentum-space methods, including the non-trivial x dependence.

APPENDIX C: PROBLEMS

P1: Complete the proof that the $\phi_n(p)$ for the ISW are orthonormal by explicit calculation of $\langle\phi_m|\phi_n\rangle$, completing the steps in Sec. III, and making use of identities such as Eqn. (47).

P2: Using results from any quantum mechanics textbook, evaluate the Wigner distribution for the ground state and first excited state of the simple harmonic oscillator. **Answer**: The harmonic oscillator eigenstates can be written in the form

$$u_n(z) = A_n H_n(z) e^{-z^2/2} \quad \text{where} \quad A_n \equiv \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \quad \text{and} \quad z \equiv \frac{x}{b} = \frac{x}{\sqrt{\hbar/m\omega}} \tag{C1}$$

and the $H_n(z)$ are the Hermite polynomials of order n. The Wigner functions for $n = 0, 1$ are then given by

$$P_W^{(0)}(x,p) = \frac{1}{\pi\hbar} e^{-\rho^2} \tag{C2}$$

$$P_W^{(1)}(x,p) = \frac{1}{\pi\hbar} e^{-\rho^2}(2\rho^2 - 1) \tag{C3}$$

where

$$\rho^2 \equiv \frac{x^2}{b^2} + \frac{b^2 p^2}{\hbar^2} \,. \tag{C4}$$

This is one of the simpler results which shows explicitly that the Wigner function need not be positive-definite. Note that the first of these results is consistent with the expression for the free-particle Gaussian in Eqn. (18) for vanishing x_0, p_0, and t. The Wigner function for arbitrary n has been evaluated (see, *e.g.*, Ref. [7]) with the result that

$$P_W^{(n)} = \frac{(-1)^n}{\pi\hbar} e^{-\rho^2} L_n(2\rho^2) \tag{C5}$$

where $L_n(z)$ are the Laguerre polynomials.

P3: Show how the phase-space plot for the 1-D harmonic oscillator can be used to generate the classical probability densities for position- and momentum-variables, namely, show how one obtains

$$P_{CL}(x) = \frac{1}{\pi\sqrt{x_A^2 - x^2}} \qquad \text{and} \qquad P_{CL}(p) = \frac{1}{\pi\sqrt{p_A^2 - p^2}} \,. \tag{C6}$$

Partial answer: The classical probability distribution from Eqn. (50) can be written in the form

$$P_{CL}(x,p) \quad \propto \quad \delta(p(x)^2 - p^2) = \frac{1}{2p(x)}\left[\delta(p(x) - p) + \delta(p(x) + p)\right] \tag{C7}$$

where

$$p(x) = \sqrt{2mE - m^2\omega^2 x^2} = m\omega\sqrt{x_A^2 - x^2} \,. \tag{C8}$$

Integration of Eqn. (C7) over dp then gives

$$P_{CL}(x) \propto \frac{1}{p(x)} \propto \frac{1}{\sqrt{x_A^2 - x^2}} \tag{C9}$$

which when properly normalized (integrated over the classically allowed interval $(-x_A, +x_A)$) gives the first term in Eqn. (C6).

P4: Using either the momentum- or position-space wavefunctions in Eqns. (14) or (15) for the free-particle Gaussian wave packet, show that the Wigner function is of the form in Eqn. (18).

P5: Complete the proof that the result for the Wigner function for eigenstates of the ISW in Eqn. (60) can be obtained using momentum-space wavefunctions, using the techniques in Appendix B.

P6: Use the results in Eqn. (30) and (63) to write down the time-dependent Wigner function for the simple two-state system in the infinite well

$$\psi(x,0) = \frac{1}{\sqrt{2}}\left[u_1(x) + u_2(x)\right] \tag{C10}$$

and generate plots of $P_W^{(\psi)}(x,p;t)$ for various times. Compare the results to standard images of the position-space and momentum-space probability densities for this problem [48], [49]. What is the only time scale associated with this system? and does it have anything to do with a classical periodicity?